Recall: A set $A \subseteq \mathbb{R}$ has measure zero if $\forall\, \varepsilon > 0$ $\exists$ open intervals $\{O_i\}_{i=1}^{\infty}$ with $A \subseteq \bigcup_{i=1}^{\infty} O_i$ and

$$\sum_{i=1}^{\infty} \ell(O_i) < \varepsilon.$$

Lemma: If $\{A_n\}_{n=1}^{\infty}$ is a sequence of measure-zero subsets of $\mathbb{R}$, then

$$A = \bigcup_{n=1}^{\infty} A_n$$ is also measure-zero.

proof: Since A_n is measure-zero $\forall\, n \in \mathbb{N}$, if $\varepsilon > 0$, $\exists$ open intervals $\{O_{n_i}\}_{i=1}^{\infty}$ such that

$$A_n \subseteq \bigcup_{i=1}^{\infty} O_{n_i} \quad \text{and} \quad \sum_{i=1}^{\infty} \ell(O_{n_i}) < \frac{\varepsilon}{2^n}.$$

Consider $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} O_{n_i}$

$$\supseteq \bigcup_{n=1}^{\infty} A_n = A.$$

Then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(O_{n_i})$$

$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon, \text{ so done.}$$

Except for the question of whether the iterated sum is well-defined!

But it is – see section 2.8. In particular, the manner in which the operations are undertaken is irrelevant. $\square$

Definition: (α-continuity)

Let $f: [a,b] \to \mathbb{R}$. If $x \in (a,b)$, we say f is **α-continuous** at x for α a fixed positive number if $\exists$ $\delta > 0$ such that $\forall$ $y, z \in (x-\delta, x+\delta)$,

$$\boxed{|f(y) - f(z)| < \alpha}$$

Note: α-continuity for a given α at $x \in [a,b]$ does **not** imply continuity at x.

Example: $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$$

is α-continuous $\forall\, \alpha > 1$ for any $x \in (0,1)$.

Notation: (D_α, D)

Let $f : [a,b] \to \mathbb{R}$, $\alpha > 0$.

$$D_\alpha = \{ x \in [a,b] \mid f \text{ is not } \alpha\text{-continuous at } x \}$$

$$D = \{ x \in [a,b] \mid f \text{ is not continuous at } x \}$$

Observations:

1) $\alpha_1 < \alpha_2 \Rightarrow D_{\alpha_2} \subseteq D_{\alpha_1}$

2) $D_\alpha \subseteq D \quad \forall \ \alpha > 0$

Lemma 1: (discontinuities)

$$D = \bigcup_{n=1}^{\infty} D_{1/n}$$

proof: From observations,

$D_{1/n} \subseteq D \ \forall\, n \in \mathbb{N}$, so

$\bigcup_{n=1}^{\infty} D_{1/n} \subseteq D$. Now let

$x \in D$. We want to show

$x \in \bigcup_{n=1}^{\infty} D_{1/n}$.

Since $x \in D$, f is not continuous at x. So $\exists\ \varepsilon > 0$ such that $\forall\ \delta > 0$ there exists a y with $|x-y| < \delta$ and $|f(x)-f(y)| \geq \varepsilon$.

Therefore, f is not ε-continuous ($\alpha = \varepsilon$) at x.

Take $n \in \mathbb{N}$, $\frac{1}{n} < \varepsilon$.

Then $x \in D_\varepsilon \subseteq D_{1/n}$

$\Rightarrow \quad x \in \bigcup_{n=1}^{\infty} D_{1/n}$, so

$$D = \bigcup_{n=1}^{\infty} D_{1/n}. \qquad \square$$

Lemma 2: D_α is closed $\forall \alpha > 0$.

proof: Equivalent to D_α^c

open.

$D_\alpha^c = \{x \in (a,b) \mid f \text{ is } \alpha\text{-continuous at } x\}$.

Let $\alpha > 0$. Choose $x \in D_\alpha^c$.

Then $\exists\ \delta > 0$ such that

$\forall\ y, z \in (x-\delta, x+\delta)$,

$|f(y) - f(z)| < \alpha$.

Let $y \in (x-\delta, x+\delta)$.

Let $\delta_y = \min\{y-(x-\delta), x+\delta-y\}$.

Then if $z, w \in (y-\delta_y, y+\delta_y)$
$\subseteq (x-\delta, x+\delta)$,

so $|f(z)-f(w)| < \alpha$

$\Rightarrow$ f is α-continuous $\forall$
$y \in (x-\delta, x+\delta) \Rightarrow$

D_α^c is open $\Rightarrow D_\alpha$ closed. $\square$

Definition: (uniform α-continuity)

Let $f: [a,b] \to \mathbb{R}$, $\alpha > 0$.

Let $S \subseteq [a,b]$. Then f is

uniformly α-continuous on

S if $\exists\ \delta > 0$ such that

$\forall\ x \in S$, f is α-continuous

at x.

Lemma 3: (α-continuity and compactness)

Let $K \subseteq [a,b]$ be compact. Then if $\alpha > 0$ and $f : [a,b] \to \mathbb{R}$ is α-continuous $\forall\, x \in K$, f is uniformly α-continuous on K.

proof: Since f is α-continuous $\forall\, x \in K$, $\exists\, \delta_x > 0$ such that $\forall\, y, z \in (x - \delta_x, x + \delta_x)$, $|f(y) - f(z)| < \alpha$.

Trivially,

$$K \subseteq \bigcup_{x \in K} (x-\delta_x, x+\delta_x).$$

Since K is compact, $\exists$

$x_1, x_2, \ldots, x_n \in K$,

$$K \subseteq \bigcup_{i=1}^{n} (x_i-\delta_{x_i}, x_i+\delta_{x_i}).$$

Let $y_1, y_2, \cdots, y_{2n}$ be the

endpoints of the intervals

$(x_i-\delta x_i, x_i+\delta x_i)$.

Let $\delta = \dfrac{\min\limits_{i,j} |y_i - y_j|}{2}$.

Then this δ will work $\forall\ x \in K$ since $\exists\ i,\ 1 \leq i \leq n,$

$(x-\delta, x+\delta) \subseteq (x_i - \delta_{x_i}, x_i + \delta_{x_i})$.

$\square$

Theorem: (Lebesgue)

Let $f: [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ if and only if D has measure zero.

proof: Next class